# Quantum Field Theory

## Set 8: solutions

## Notation

The Noether theorem states the following: given a theory described by the Lagrangian density $\mathcal{L}[\phi_a, \partial_\mu \phi_a]$, where $\phi_a$ is a generic field, and a symmetry of this Lagrangian acting on coordinates and fields as follows at the infinitesimal level (with some transformation labelled by a set of parameters $\{\alpha^i\}$)

$$\begin{aligned}
x^\mu &\longrightarrow x'^\mu \simeq x^\mu - \epsilon^\mu_i \alpha^i, \\
\phi_a(x) &\longrightarrow \phi'_a(x') = D[\phi]_a(x) \simeq \phi_a(x) + \alpha^i \varepsilon_{ai}(x) \simeq \phi_a(x) + \alpha^i \varepsilon_{ai}(x') \\
&\qquad\qquad\quad \simeq \phi_a(x') + \alpha^i \epsilon^\mu_i \partial_\mu \phi_a(x') + \alpha^i \varepsilon_{ai}(x') \\
&\qquad\qquad\quad \equiv \phi_a(x') + \alpha^i \Delta_{a\,i}(x'),
\end{aligned}$$

then the four vector defined by

$$J^\mu_i = \sum_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \Delta_{a\,i} - \epsilon^\mu_i \mathcal{L}$$

satisfies $\partial_\mu J^\mu_i = 0$ when one uses the equation of motion for the field $\phi_a$. For clearness, note that the easiest way to compute $\Delta$ is

$$\begin{aligned}
\varepsilon_{a\,i}(x')\alpha^i &= \phi'_a(x') - \phi_a(x), \\
\Delta_{a\,i}(x')\alpha^i &= \phi'_a(x') - \phi_a(x'), \\
\Delta_{a\,i}(x)\alpha^i &= \phi'_a(x) - \phi_a(x).
\end{aligned}$$

Note also that when the symmetry is such that $x' = x$, then $\Delta_{ai} = \varepsilon_{ai}$.

## Exercise 1

The usual classical mechanics can be regarded as a $0+1$ dimensional field theory. The coordinates that describe a classical system are functions of time and therefore can be thought of as fields:

| $3+1$ | $0+1$ |
|---|---|
| $\vec{x}, t$ | $t$ |
| $\phi_a(\vec{x}, t)$ | $\vec{q}_a(t)$ |

where $\vec{q}_a(t)$ are coordinates and can be in general a three-dimensional vector (note that the three dimensions are those of the physical space where the system evolves). Since the classical mechanics is a trivial example of field theory, one can directly apply the formulas obtained from the Noether theorem and get several currents corresponding to different symmetries. For example a Lagrangian of the form

$$L = \sum_a \frac{m_a}{2} |\dot{\vec{q}}_a|^2 - \sum_{a \neq b} V(|\vec{q}_a - \vec{q}_b|)$$

has the following symmetries:

- time translation: $t' = t - \alpha$, $\vec{q}\,'_a(t') = \vec{q}_a(t)$;
- collective coordinates translation: $t' = t$, $\vec{q}\,'_a = \vec{q}_a + \vec{\alpha}$;
- coordinates rotation: $t' = t$, $q'^i_a = \mathcal{R}^i_j q^j_a$.

In the case of time translation one gets

$$\begin{aligned}
&t' = t - \alpha \implies \epsilon = 1, \\
&\vec{q}_a'(t') - \vec{q}_a(t) = 0 \implies \vec{\varepsilon}_a = 0, \\
&\vec{q}_a'(t) - \vec{q}_a(t) = \alpha\dot{\vec{q}}_a \implies \vec{\Delta}_a = \dot{\vec{q}}_a,
\end{aligned}$$

where we have used the vector notation for $\varepsilon_a$ and $\Delta_a$ since there is one of them for any component of the coordinate vector. And hence the current is

$$J^0 = \sum_a \frac{\partial L}{\partial \dot{\vec{q}}_a} \cdot \vec{\Delta}_a - L = \sum_a \vec{\pi}_a \cdot \dot{\vec{q}}_a - L = H.$$

The Noether current associated to the invariance under time translation coincides with the Hamiltonian and the condition of null divergence is $\partial_0 J^0 = \frac{\partial H}{\partial t} = 0$, so the Hamiltonian is conserved.

Analogously one can consider translations along some direction $\hat{n}$:

$$\begin{aligned}
&t' = t \implies \epsilon = 0, \\
&\vec{q}_a'(t') - \vec{q}_a(t) = \alpha\hat{n} \implies \vec{\varepsilon}_a = \hat{n}, \\
&\vec{q}_a'(t) - \vec{q}_a(t) = \alpha\hat{n} \implies \vec{\Delta}_a = \hat{n}, \\
&J^0 = \sum_a \frac{\partial L}{\partial \dot{\vec{q}}_a} \cdot \vec{\Delta}_a = \sum_a \vec{\pi}_a \cdot \hat{n} = \vec{P} \cdot \hat{n}.
\end{aligned}$$

Therefore the invariance under translation along some direction implies the conservation of the conjugate momentum in that direction.

Finally, consider the invariance under rotation around a direction (take the $z$ direction for concreteness):

$$\begin{aligned}
&t' = t \implies \epsilon = 0, \\
&q_a'^x(t') = q_a^x(t) + \alpha q_a^y(t) \implies q_a'^x(t) - q_a^x(t) = \alpha q_a^y(t) \implies \Delta_a^x = q_a^y(t), \\
&q_a'^y(t') = -\alpha q_a^x(t) + q_a^y(t) \implies q_a'^y(t) - q_a^y(t) = -\alpha q_a^x(t) \implies \Delta_a^y = -q_a^x(t), \\
&q_a'^z(t') = q_a^z(t) \implies \Delta_a^z = 0, \\
&J^0 = \sum_a \left( \frac{\partial L}{\partial \dot{q}_a^x}\Delta_a^x + \frac{\partial L}{\partial \dot{q}_a^y}\Delta_a^y \right) = \sum_a \left(\pi_a^x q_a^y - \pi_a^y q_a^x\right) = \sum_a (\vec{\pi}_a \wedge \vec{q}_a)^z.
\end{aligned}$$

Therefore the invariance under rotation around some direction implies the conservation of the angular momentum in that direction.

## Exercise 2

Consider the example of a free complex scalar field $\phi = \phi_1 + i\phi_2$. The Lagrangian density reads

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2\, \phi^\dagger \phi.$$

It's straightforward to show that the transformation given in the text is a symmetry of the theory: the transformed Lagrangian is in fact

$$\mathcal{L}' = \partial_\mu \phi^\dagger e^{-i\alpha} \partial^\mu \phi e^{i\alpha} - m^2 \phi^\dagger e^{-i\alpha} \phi e^{i\alpha},$$

which is equal to $\mathcal{L}$ because $\alpha$ is a *global* parameter (it's not a function of coordinates).
To find the Noether current it is useful to write the transformation at the infinitesimal level:

$$\begin{aligned}
&x' = x \implies \epsilon^\mu = 0, \\
&\phi'(x') = e^{i\alpha}\phi(x) \sim (1 + i\alpha)\phi(x) \implies \varepsilon_\phi = i\phi, \\
&\phi'^\dagger(x') = e^{-i\alpha}\phi^\dagger(x) \sim (1 - i\alpha)\phi^\dagger(x) \implies \varepsilon_{\phi^\dagger} = -i\phi^\dagger.
\end{aligned}$$

Here the indeces $a$, $i$ appearing in the general definition and labeling respectively the fields and the parameters of the transformation assume the values

$$\begin{aligned} &a = \phi,\, \phi^\dagger, \\ &i = 1, \qquad \text{since only the phase } \alpha \text{ is present.} \end{aligned}$$

The Noether current reads

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Delta_\phi + \Delta_{\phi^\dagger}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)} = \partial^\mu \phi^\dagger (i\phi) + (-i\phi^\dagger)\partial^\mu \phi,$$

and the divergence gives

$$-i\partial_\mu J^\mu = \Box \phi^\dagger\, \phi - \phi^\dagger \Box \phi = m^2 \phi^\dagger \phi - \phi^\dagger (m^2)\phi = 0,$$

where in the last equality we have used the equations of motion for $\phi$ and $\phi^\dagger$, namely $\Box\phi = m^2\phi$ and $\Box\phi^\dagger = m^2\phi^\dagger$. This is an important feature of the Noether current: its divergence vanishes only if one makes explicit use of the equations of motion for the fields. The above behavior is somehow expected because in the derivation of the current one neglects terms that are proportional to the Euler-Lagrange equation:

$$\int d^4x'\; \mathcal{L}[\phi'](x') - \int d^4x\; \mathcal{L}[\phi](x) \sim \int d^4x\; \alpha^i \partial_\mu J^\mu_i + \int d^4x \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} - \frac{\partial \mathcal{L}}{\partial \phi_a}\right)\Delta_{ai}\alpha^i.$$

If the transformations define a symmetry the l.h.s. has to be identically zero and therefore one deduces the vanishing of the divergence of the current once the Euler-Lagrange equation are satisfied.

If we add to the Lagrangian an interaction term of the form $V(\phi^\dagger\phi)$, the symmetry is preserved. One can check that current retains the same form, only the equation of motion changes:

$$\Box\phi - \left(m^2 + V'(\phi^\dagger\phi)\right)\phi = \Box\phi^\dagger - \left(m^2 + V'(\phi^\dagger\phi)\right)\phi^\dagger = 0 \tag{1}$$

## Exercise 3

Given a symmetry acting on coordinates and fields as follows

$$\begin{aligned} &x^\mu \longrightarrow x'^\mu = f^\mu(x) \simeq x^\mu - \epsilon^\mu_i \alpha^i, \\ &\phi_a(x) \longrightarrow \phi'_a(x') = D[\phi]_a(f^{-1}(x')) \simeq \phi_a(x) + \alpha^i \varepsilon[\phi]_{ai}(x) = \phi_a(x') + \alpha^i \Delta[\phi]_{ai}(x'), \end{aligned}$$

Noether's current is defined as

$$J^\mu_i = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\,\Delta_{ai} - \epsilon^\mu_i \mathcal{L},$$

satisfying $\partial_\mu J^\mu_i = 0$. As a consequence, one defines the conserved Noether's charge associated to the symmetry

$$Q_i = \int d^3x\; J^0_i = \int d^3x \left(\pi_a\, \Delta_{ai} - \epsilon^0_i \mathcal{L}\right), \qquad \dot{Q}_i = 0,$$

where $\pi_a$ is the momentum conjugate to the field $\phi_a$. One can use this charge to define a generator of the symmetry transformations in the following sense:

$$\delta_0\phi(x) = \phi'(x) - \phi(x) = \Delta_{ai}(x)\alpha^i = \{Q_i, \phi_a(x)\}\,\alpha^i.$$

Indeed substituting the expression for Noether's charge and recalling the definition of the Poisson brackets ( see the solutions of Set 2) one gets

$$\begin{aligned} \{Q_i, \phi_a(x)\} &= \int d^3y \left(\{\pi_b(y), \phi_a(x)\}\,\Delta_{bi}(y) + \pi_b(y)\,\{\Delta_{bi}(y), \phi_a(x)\} - \epsilon^0_i\,\{\mathcal{L}(y), \phi_a(x)\}\right) \\ &= \int d^3y \left(\delta_{ab}\delta^3(x-y)\Delta_{bi}(y) + \pi_b(y)\delta^3(x-y)\frac{\partial\,\Delta_{bi}}{\partial\,\pi_a}(x) - \epsilon^0_i \delta^3(x-y)\frac{\partial\,\mathcal{L}}{\partial\,\pi_a}(x)\right) \\ &= \Delta_{ai}(x) + \pi_b(x)\frac{\partial\left(\varepsilon_{bi}[\phi] + \epsilon^\mu_i \partial_\mu \phi_b\right)}{\partial\,\pi_a}(x) - \epsilon^0_i\,\frac{\partial\,\mathcal{L}}{\partial\,\partial_0\phi_b}\,\frac{\partial(\partial_0\phi_b)}{\partial\pi_a}(x) \\ &= \Delta_{ai}(x) + \pi_b(x)\epsilon^0_i\frac{\partial\,(\partial_0\phi_b)}{\partial\,\pi_a}(x) - \epsilon^0_i \pi_b(x)\,\frac{\partial(\partial_0\phi_b)}{\partial\pi_a}(x) = \Delta_{ai}(x), \end{aligned}$$

where we have used the fact that in the Hamiltonian formalism the independent variables are $\phi_a$ and $\pi_a$, while the spatial derivatives of the field, $\partial_i\phi_a$, are considered non independent.
The Noether's charges, as functions of the fields and their conjugate momentum, can be thought of as generators in the Poisson brackets formalism, in complete analogy with analytical mechanics. The label *generators* has however a deeper meaning. Since the transformations reflect the action of an element of the Lie symmetry group, one can regard this charges as a realization of the Lie algebra on the space of fields. The operation under which these generators must be closed are the Poisson brackets.

Consider the explicit example of translations:

$$
\begin{aligned}
x^\mu &\longrightarrow x'^\mu = x^\mu - a^\mu = x^\mu - \delta^\mu_\nu a^\nu,\\
\phi_a(x) &\longrightarrow \phi'_a(x') = \phi_a(x) = \phi_a(x') + a^\nu \partial'_\nu \phi_a(x').
\end{aligned}
$$

The associated Noether's current is called *Energy-Momentum Tensor* and can be computed as usual:

$$
\begin{aligned}
T^\mu_\nu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\Delta_{a\nu} - \epsilon^\mu_\nu \mathcal{L}\\
&= \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\partial_\nu\phi_a - \delta^\mu_\nu \mathcal{L}.
\end{aligned}
$$

The Noether's charge in the present case is a four vector:

$$
P_\nu = \int d^3x \left(\pi_a\,\partial_\nu\phi_a - \delta^0_\nu\mathcal{L}\right) = \left\{ \begin{array}{ll} \int d^3x\,(\pi_a\,\partial_0\phi_a - \mathcal{L}) = H & \text{for } \nu = 0,\\ \int d^3x\,(\pi_a\partial_i\phi_a) & \text{for } \nu = i. \end{array}\right.
$$

The zero component of the Noether's charge corresponds to the energy of the system, therefore the complete four vector is the conserved four momentum of the system. Finally the infinitesimal transformation can be obtained using the Poisson brackets formalism:

$$
\begin{aligned}
&\{H,\phi_a(x)\} = \dot{\phi}_a(x),\\
&\{P_i,\phi_a(x)\} = \int d^3y\,\{\pi_b(y)\partial_i\phi_b(y),\phi_a(x)\} = \int d^3y\ \delta^3(x-y)\delta_{ab}\partial_i\phi_b(y) = \partial_i\phi_a(x),
\end{aligned}
$$

which confirms the interpretation of $P_i$ as spatial momentum. One can repeat the procedure for the Lorentz transformations, assuming the theory to be Poincaré invariant, (and for simplicity assume again scalar fields):

$$
\begin{aligned}
x^\mu &\longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu \simeq x^\mu - w^\mu{}_\nu x^\nu = x^\mu - w^{\alpha\beta}(\eta_{\beta\nu}\delta^\mu_\alpha - \eta_{\alpha\nu}\delta^\mu_\beta)x^\nu,\\
\phi_a(x) &\longrightarrow \phi'_a(x') = \phi_a(x) = \phi_a(x') + w^{\alpha\beta}(\eta_{\beta\nu}\delta^\mu_\alpha - \eta_{\alpha\nu}\delta^\mu_\beta)x'^\nu\partial'_\mu\phi_a(x').
\end{aligned}
$$

The associated Noether's current is:

$$
\begin{aligned}
M^\mu_{\alpha\beta} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\Delta_{a\alpha\beta} - \epsilon^\mu_{\alpha\beta}\mathcal{L}\\
&= \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_a)}\,(x_\alpha\partial_\beta\phi_a - x_\beta\partial_\alpha\phi_a) - (x_\alpha\delta^\mu_\beta - x_\beta\delta^\mu_\alpha)\mathcal{L}\\
&= x_\alpha T^\mu{}_\beta - x_\beta T^\mu{}_\alpha,
\end{aligned}
$$

where $T^\mu{}_\beta$ is the energy momentum tensor associated to spacetime translation invariance. The conservation of the current reads:

$$
\partial_\mu\, M^\mu_{\alpha\beta} = T_{\alpha\beta} - T_{\beta\alpha} = 0,
$$

so the invariance under Lorentz transformations, together with the one under translations, implies the symmetry of the energy momentum tensor in the two indices (for a scalar theory). Notice in fact that we have assumed that the theory is Poincaré invariant, so the energy momentum tensor is by itself conserved. The Noether's charge in the present case is a Lorentz tensor:

$$
Q_{\alpha\beta} = \int d^3x \left(x_\alpha T^0{}_\beta - x_\beta T^0{}_\alpha\right) = \int d^3x \left(x_\alpha(\pi_a\partial_\beta\phi_a - \delta^0_\beta\mathcal{L}) - x_\beta(\pi_a\partial_\alpha\phi_a - \delta^0_\alpha\mathcal{L})\right).
$$

To see the meaning of this charge one can compute the Poisson bracket of its $ij-$component with the field $\phi_a(x)$:

$$
\{Q_{ij},\phi_a(x)\} = \int d^3y\,(y_i\,\{\pi_b(y),\phi_a(x)\}\,\partial_j\phi_b(y) - y_j\,\{\pi_b(y),\phi_a(x)\}\,\partial_i\phi_b(y)) = (x_i\partial_j - x_j\partial_i)\phi_a(x).
$$

This charge is thus the angular momentum tensor of the system.

## A note on functional derivatives

Consider a functional $F[\phi]$ defined as

$$F[\phi] = \int d^4x \; f(\phi, \partial\phi, x).$$

Given a small variation of the field, $\phi(x) \longrightarrow \phi(x) + \delta\phi(x)$, then the change in $F$ is

$$\delta F = \int d^4x \; \frac{\delta F}{\delta \phi(x)} \delta\phi(x) + O(\delta\phi^2),$$

which generalizes the usual definition in the discrete case

$$\delta F = \sum_i \frac{\partial F}{\partial q_i} \delta q_i + O(\delta q_i^2).$$

In terms of the integrand, the variation $\delta F$ is

$$\delta F = \int d^4x \left( \frac{\partial f(x)}{\partial \phi(x)} \delta\phi(x) + \frac{\partial f(x)}{\partial(\partial_\mu \phi(x))} \partial_\mu \delta\phi(x) \right),$$

which, up to boundary terms, can be rewritten as

$$\delta F = \int d^4x \left( \frac{\partial f(x)}{\partial \phi(x)} - \partial_\mu \frac{\partial f(x)}{\partial(\partial_\mu \phi(x))} \right) \delta\phi(x).$$

In the previous equations we have simply renamed $f(\phi, \partial\phi, x) \equiv f(x)$. Note that the symbol $\partial$ stands for *ordinary* derivative. Identifying this equation and the second, one gets the definition of functional derivative $\delta$ as

$$\frac{\delta F}{\delta \phi(x)} = \frac{\partial f(x)}{\partial \phi(x)} - \partial_\mu \frac{\partial f(x)}{\partial(\partial_\mu \phi(x))}.$$

An example in which this definition is used is the Euler-Lagrange equation

$$\frac{\delta S}{\delta \phi(x)} = 0 \quad \Longleftrightarrow \quad \frac{\partial \mathcal{L}(x)}{\partial \phi(x)} - \partial_\mu \frac{\partial \mathcal{L}(x)}{\partial(\partial_\mu \phi(x))} = 0.$$

In particular one can consider $F[\phi] = \phi(y)$, so that $f(x) = \phi(x)\delta^4(x-y)$ and

$$\frac{\delta \phi(y)}{\delta \phi(x)} = \frac{\partial(\phi(x)\delta^4(x-y))}{\partial \phi(x)} = \delta^4(x-y),$$

which is the generalization of the discrete relation $\partial q_i / \partial q_j = \delta^i_j$.

In the hamiltonian formalism time is distinguished from the spatial coordinates, so one has integrals over $d^3x$ rather than over $d^4x$. Given a functional $F$ of fields and conjugate momenta

$$F[\phi, \pi](t) = \int d^3x \; f(\phi, \partial_i\phi, \pi, x, t),$$

its change due to variations $\delta\phi(x,t)$ and $\delta\pi(x,t)$ is

$$\begin{aligned}
\delta F(t) &= \int d^3x \; \left( \frac{\delta F(t)}{\delta \phi(x,t)} \delta\phi(x,t) + \frac{\delta F(t)}{\delta \pi(x,t)} \delta\pi(x,t) \right) + O(\delta\phi^2, \delta\pi^2) \\
&= \int d^3x \; \left( \frac{\partial f(x,t)}{\partial \phi(x,t)} \delta\phi(x,t) - \partial_i \frac{\partial f(x,t)}{\partial(\partial_i \phi(x,t))} \delta\phi(x,t) + \frac{\partial f(x,t)}{\delta \pi(x,t)} \delta\pi(x,t) \right),
\end{aligned}$$

so that

$$\begin{aligned}
\frac{\delta F(t)}{\delta \phi(x,t)} &= \frac{\partial f(x,t)}{\partial \phi(x,t)} - \partial_i \frac{\partial f(x,t)}{\partial(\partial_i \phi(x,t))}, \\
\frac{\delta F(t)}{\delta \pi(x,t)} &= \frac{\partial f(x,t)}{\partial \pi(x,t)}.
\end{aligned}$$

Note that in the last steps we have neglected boundary terms and used the shorthand notation $f(\phi, \partial_i\phi, \pi, x, t) \equiv f(x,t)$. Particularly relevant cases are $F[\phi,\pi](t) = \pi(y,t)$, corresponding to $f(x,t) = \pi(x,t)\delta^3(x-y)$, which yields

$$\frac{\delta\pi(y,t)}{\delta\pi(x,t)} = \frac{\partial(\pi(x,t)\delta^3(x-y))}{\partial\pi(x,t)} = \delta^3(x-y),$$

and $F[\phi,\pi](t) = \phi(y,t)$, or $f(x,t) = \phi(x,t)\delta^3(x-y)$, which gives

$$\frac{\delta\phi(y,t)}{\delta\phi(x,t)} = \frac{\partial(\phi(x,t)\delta^3(x-y))}{\partial\phi(x,t)} = \delta^3(x-y).$$

Given now two functionals

$$A[\phi,\pi](t) = \int d^3x \; a(\phi, \partial_i\phi, \pi, x, t),$$
$$B[\phi,\pi](t) = \int d^3x \; b(\phi, \partial_i\phi, \pi, x, t),$$

their Poisson brackets (at constant time) are defined starting from the definition of functional derivative as

$$\begin{aligned}\{A[\phi,\pi](t), B[\phi,\pi](t)\} &= \int d^3x \left(\frac{\delta A(t)}{\delta\pi(x,t)}\frac{\delta B(t)}{\delta\phi(x,t)} - \frac{\delta B(t)}{\delta\pi(x,t)}\frac{\delta A(t)}{\delta\phi(x,t)}\right) \\ &= \int d^3x \left[\frac{\partial a(x,t)}{\partial\pi(x,t)}\left(\frac{\partial b(x,t)}{\partial\phi(x,t)} - \partial_i\frac{\partial b(x,t)}{\partial(\partial_i\phi(x,t))}\right) - \frac{\partial b(x,t)}{\partial\pi(x,t)}\left(\frac{\partial a(x,t)}{\partial\phi(x,t)} - \partial_i\frac{\partial a(x,t)}{\partial(\partial_i\phi(x,t))}\right)\right].\end{aligned}$$

For the particular case $A = \pi(z,t)$, $B = \phi(w,t)$, one finds

$$\{\pi(z,t), \phi(w,t)\} = \int d^3x \frac{\delta\pi(z,t)}{\delta\pi(x,t)}\frac{\delta\phi(w,t)}{\delta\phi(x,t)} = \int d^3x \; \delta^3(x-z)\; \delta^3(w-x) = \delta^3(z-w).$$

## Exercise 4

In addition to the usual transformation properties under the Poincaré group (that define scalars, vectors, spinors, ...), any given field can have non trivial transformation properties under the action of additional symmetries. In the present case we consider a pair of Lorentz scalars $\phi_1$ and $\phi_2$, which are assumed to transform in the representation $j = 1/2$ of an $SU(2)$ group. This *isospin* symmetry acts on fields as

$$\begin{aligned}\Phi(x) &\longrightarrow \Phi'(x') = \mathcal{U}\Phi(x), \qquad \Phi^\dagger(x) \longrightarrow \Phi'^\dagger(x') = \Phi^\dagger(x)\mathcal{U}^\dagger, \\ \phi_a(x) &\longrightarrow \phi'_a(x') = \mathcal{U}_a{}^b\phi_b(x), \qquad \phi^{*a}(x) \longrightarrow \phi'^{*a}(x') = \phi^{*b}(x)(\mathcal{U}^\dagger)_b{}^a,\end{aligned}$$

where $\mathcal{U}$ is a $SU(2)$ element and we have defined the complex conjugate fields with the index up to recall that they transform with the hermitian conjugate matrix (they are a row instead that a column). The Lagrangian density

$$\mathcal{L} = \partial_\mu\Phi^\dagger\partial^\mu\Phi - V\left(\Phi^\dagger\Phi\right)$$

is invariant because the two doublets $\Phi$ and $\Phi^\dagger$ are contracted in such a way to form an invariant under $SU(2)$ transformations (that is to say an object transforming in the $j = 0$ representation):

$$\Phi^\dagger\Phi \longrightarrow \Phi'^\dagger\Phi' = \Phi^\dagger\underbrace{\mathcal{U}^\dagger\mathcal{U}}_{1}\Phi,$$
$$\partial_\mu\Phi^\dagger\partial^\mu\Phi \longrightarrow \partial_\mu\Phi'^\dagger\partial^\mu\Phi' = \partial_\mu(\Phi^\dagger\mathcal{U}^\dagger)\partial^\mu(\mathcal{U}\Phi) = \partial_\mu\Phi^\dagger\underbrace{\mathcal{U}^\dagger\mathcal{U}}_{1}\partial^\mu\Phi,$$

where we have used the definition of $SU(2)$ matrices. Since the invariance of the Lagrangian density has been proved for a generic potential being function only of the $SU(2)$ invariant combination $\Phi^\dagger\Phi$, one can go on and compute the Noether's current associated to this symmetry. Using the exponential representation of the $SU(2)$ matrices in terms of the generators (which we showed to be the Pauli matrices divided by two in the $j = 1/2$ representation)

$$\phi'_a(x') = \left(e^{i\alpha^i\frac{\sigma^i}{2}}\right)_a{}^b\phi_b(x) \simeq \phi_a(x) + i\alpha^i\left(\frac{\sigma^i}{2}\right)_a{}^b\phi_b \equiv \phi_a(x) + \Delta_{\phi a}{}^i(x)\alpha^i,$$
$$\phi'^{*c}(x') = \phi^{*b}(x)\left(e^{-i\alpha^i\frac{\sigma^i}{2}}\right)_b{}^c \simeq \phi^{*c}(x) - i\alpha^i\phi^{*b}(x)\left(\frac{\sigma^i}{2}\right)_b{}^c \equiv \phi^{*c}(x) + \Delta_{\phi^*}{}^{ci}(x)\alpha^i,$$

where we have defined $\Delta_\phi$ and $\Delta_{\phi^*}$ as the variations of $\phi$ and $\phi^*$, respectively. Note that in this case there is no variation of the position $x$, so the total variation of the fields coincides with the variation at fixed coordinate, hence the use of the name $\Delta(x)$ instead of $\varepsilon(x)$, see Solution8. The Noether's current then reads

$$\begin{aligned} J^{i\,\mu} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \Delta_{\phi a}^{\ \ i} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{*a})} \Delta_{\phi^*}^{\ \ ai} = i\partial^\mu \phi^{*a} \left(\frac{\sigma^i}{2}\right)_a^{\ b} \phi_b - i\partial^\mu \phi_c \phi^{*b} \left(\frac{\sigma^i}{2}\right)_b^{\ c} \\ &= \frac{i}{2}\partial^\mu \phi^{*a} (\sigma^i)_a^{\ b} \phi_b - \frac{i}{2}\phi^{*b} (\sigma^i)_b^{\ c} \partial^\mu \phi_c = \frac{i}{2}\partial^\mu \Phi^\dagger \sigma^i \Phi - \frac{i}{2}\Phi^\dagger \sigma^i \partial^\mu \Phi. \end{aligned}$$

One has to notice the similarity of this expression with the Noether's current associated to the $U(1)$ symmetry: the previous current is its obvious generalization and the only difference is the presence of the generators between the fields in the appropriate representation. Another point to be stressed is the complete independence of the result here obtained from the explicit form of the potential $V$, since this doesn't contain derivative of the fields.

Let us now consider another set of three Lorentz scalar fields $\vec{A} = (A_1,\, A_2,\, A_3)$ and impose that under the action of $SU(2)$ they transform in the representation $j = 1$:

$$\begin{aligned} &\Phi(x) \longrightarrow \Phi'(x') = \mathcal{U}\Phi(x), \qquad \Phi^\dagger(x) \longrightarrow \Phi'^\dagger(x') = \Phi^\dagger(x)\mathcal{U}^\dagger, \\ &\vec{A}(x) \longrightarrow \vec{A}(x') = \mathcal{R}[\mathcal{U}]^{(j=1)}\vec{A}. \end{aligned}$$

Notice that the representation $j = 1$ of $SU(2)$ consists of rotations acting on a three dimensional vector space (and coincides with $SO(3)$ rotations) therefore we have collectively denoted the fields $A_i$ with a vector $\vec{A}$. The matrix $\mathcal{R}[\mathcal{U}]^{(j=1)}$ is the representative of $\mathcal{U}$ in the representation $j = 1$, that is to say the element of $SO(3)$ associated to $\mathcal{U}$. We add to the previous Lagrangian density a kinetic term for the fields $A$ and an interaction term:

$$\tilde{\mathcal{L}} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - V\left(\Phi^\dagger \Phi\right) + \frac{1}{2}\partial_\mu A^T \partial^\mu A + \lambda A_i \Phi^\dagger \sigma^i \Phi.$$

The last term of the r.h.s. is called an *interaction* term since it connects the equations of motion for the different fields:

$$\begin{aligned} \phi_a: \qquad \Box\phi^{*a} &= -\frac{\partial V}{\partial \phi_a} + \lambda A_i \phi^{*c}(\sigma^i)_c^{\ a}, \\ \phi^{*c}: \qquad \Box\phi_c &= -\frac{\partial V}{\partial \phi^{*c}} + \lambda A_i (\sigma^i)_c^{\ a}\phi_a, \\ A_i: \qquad \Box A_i &= \lambda \Phi^\dagger \sigma^i \Phi. \end{aligned}$$

One has now to check whether the part of the Lagrangian involving $A$ is invariant under $SU(2)$ transformations:

$$\partial_\mu A^T \partial^\mu A \longrightarrow \partial_\mu A'^T \partial^\mu A' = \partial^\mu A^T \underbrace{(\mathcal{R}[\mathcal{U}]^{(j=1)})^T \mathcal{R}[\mathcal{U}]^{(j=1)}}_{1} \partial_\mu A = \partial_\mu A^T \partial^\mu A,$$

because $\mathcal{R}[\mathcal{U}]^{(j=1)}$ is a matrix of $SO(3)$. The interaction term is more subtle (from now on we will omit the notation $(j = 1)$):

$$A_i' \Phi'^\dagger \sigma^i \Phi' = \mathcal{R}[\mathcal{U}]_i^{\ j} A_j \Phi^\dagger(\mathcal{U}^\dagger \sigma^i \mathcal{U})\Phi.$$

In order to complete the proof of the invariance of $\mathcal{L}$ under $SU(2)$, one should recall the definition of *adjoint* representation of $SU(2)$, that is to say the representation acting on the algebra itself. In Solution5 we have shown that, given the space of the hermitian traceless $2 \times 2$ matrices $M$ (which coincides with the Lie algebra of $SU(2)$), the action of an element $\mathcal{V}$ of $SU(2)$, defined by

$$M = x_i \sigma^i \longrightarrow \mathcal{V} M \mathcal{V}^\dagger = M' = x_i' \sigma^i,$$

induces on this space a representation which turns out to be the $j = 1$ representation, since $x_i' = \mathcal{R}[\mathcal{V}]_i^{\ j} x_j$. In particular the Pauli matrices themselves transform as a triplet:

$$\sigma^k = \delta_i^k \sigma^i \longrightarrow \mathcal{V}(\delta_i^k \sigma^i)\mathcal{V}^\dagger = (\mathcal{R}[\mathcal{V}]_i^{\ j} \delta_j^k)\sigma^i = \mathcal{R}[\mathcal{V}]_i^{\ k} \sigma^i$$

In the case at hand we have $\mathcal{U}^\dagger \sigma^i \mathcal{U} = \mathcal{R}[\mathcal{U}^\dagger]_l^{\ i} \sigma^l$, and collecting all together we obtain:

$$A_i' \Phi'^\dagger \sigma^i \Phi' \longrightarrow \underbrace{\mathcal{R}[\mathcal{U}]_i^{\ j} \mathcal{R}[\mathcal{U}^\dagger]_l^{\ i}}_{\delta_l^j} A_j \Phi^\dagger \sigma^l \Phi = A_i \Phi^\dagger \sigma^i \Phi,$$

where we have used the fact that $(\mathcal{R}[\mathcal{U}^\dagger]^{(j=1)}\mathcal{R}[\mathcal{U}]^{(j=1)})_i{}^m = \mathcal{R}[\mathcal{U}^\dagger\mathcal{U}]_i{}^m = \mathcal{R}[1]_i{}^m = \delta_i{}^m$. This completes the proof of invariance under the $SU(2)$ symmetry of the Lagrangian density $\tilde{\mathcal{L}}$.

Let us now try to write other $SU(2)$ invariant terms with dimension less or equal to four. Since $[\Phi] = [A] = 1$ not to exceed dimension four we can use either $n \leq 4$ fields or 2 derivatives and 2 fields (Lorentz invariance forbids the appearence of a single derivative). With two derivatives the only terms that can be written are the kinetic terms for $A$ and $\Phi$, already present in $\mathcal{L}$.
At dimension two, the only $SU(2)$ invariants we can build are $A^T A$ and $\Phi^\dagger\Phi$. Then the only new term we can add is:

$$\delta\mathcal{L}^{d=2} = \frac{1}{2}m_A^2(A^T A).$$

At dimension three the only possible invariant is $A_i\Phi^\dagger\sigma^i\Phi$ hence we cannot add any new term.
Finally at dimension four we can add the following term:

$$\delta\mathcal{L}^{d=4} = a(A^T A)^2$$

Notice indeed that by virtue of the identity $\sigma^i\sigma^j = \delta^{ij}1 + 2i\varepsilon^{ijk}\sigma^k$ terms of the kind $\Phi^\dagger\sigma^i\ldots\sigma^k\Phi$ can always be reduced to $\Phi^\dagger\Phi$ and $\Phi^\dagger\sigma^i\Phi$. Also we have:

$$(\Phi^\dagger\sigma^i\Phi)(\Phi^\dagger\sigma^i\Phi) = (\Phi^\dagger\Phi)^2$$

because of the identity $\sigma^i_{\alpha\beta}\sigma^i_{\gamma\delta} = 2\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta}$.